MATH 312

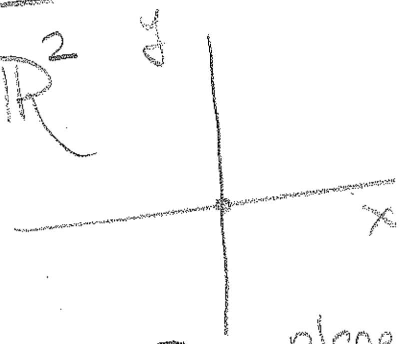
Lecture 6 : Agglutination

LINEAR COMBINATIONS & VECTOR SPACES

$\mathbb{R}$

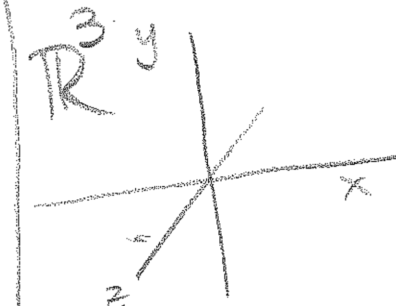
The line

$\mathbb{R}^2$



The plane

$\mathbb{R}^3$



The ...?

And we can solve LINEAR SYSTEMS in each of these three "spaces" of variables. In principle, we could do the same in $\mathbb{R}^n$! So, let's think of some fixed n, and note:

$$\mathbb{R}^n = \left\{ \underbrace{\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{\text{"vector"}} \text{ where each } a_i \text{ is in } \mathbb{R} \right\}.$$

We can:

- ADD $\vec{a} + \vec{b} = \begin{pmatrix} a_1+b_1 \\ \vdots \\ a_n+b_n \end{pmatrix}$
- SCALE $x\,\vec{a} = \begin{pmatrix} xa_1 \\ \vdots \\ xa_n \end{pmatrix}$ [x is in $\mathbb{R}$]

BUT: we can't multiply $\vec{a} \cdot \vec{b}$ unless $n = 1$

A LINEAR COMBINATION of $\vec{a}$ and $\vec{b}$ is $\underbrace{\underbrace{x\vec{a}}_{\text{scale}} + \underbrace{y\vec{b}}_{\text{scale}}}_{\text{add}}$ (x and y are scalars)

This is IMPORTANT

Eg In $\mathbb{R}^2$, consider $\vec{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
Then, a linear combination of $\vec{a}$ and $\vec{b}$ is just

$$x\begin{pmatrix} 3 \\ 5 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3x + 2y \\ 5x + y \end{pmatrix}$$

- Hmm... this could be the "LEFT SIDE" of a 2×2 system with matrix $\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}$

$$\begin{array}{cc} \begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix} \\ \uparrow \quad \uparrow \\ \vec{a} \quad \vec{b} \end{array}$$

So: The question "Is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ a lin comb. of $\vec{a}$ and $\vec{b}$?" is part of the "equivalence":

$$\begin{aligned} 3x + 2y &= 1 \\ 5x + y &= 3 \end{aligned}$$

Linear System

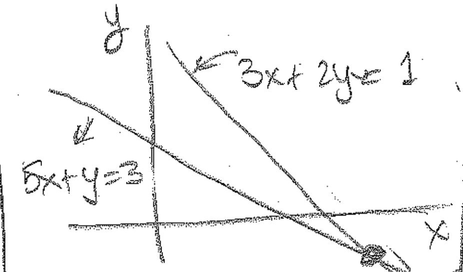
$$\begin{bmatrix} 3 & 2 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Matrix equation

SAME

$$x\begin{bmatrix} 3 \\ 5 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Linear combination



Line Intersections

Note We can also define linear combinations of more than two vectors! It's "obvious":

$$x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_m\vec{a_m}$$

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eg:
- $x\begin{pmatrix}1\\2\end{pmatrix} + y\begin{pmatrix}3\\1\end{pmatrix} + z\begin{pmatrix}0\\4\end{pmatrix}$
  is a linear combination of three vectors in $\mathbb{R}^2$,
- $x\begin{pmatrix}1\\3\\5\\9\end{pmatrix} + y\begin{pmatrix}2\\1\\7\\0\end{pmatrix} + z\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$
  is a lin comb of three vectors in $\mathbb{R}^4$.

Def (very, very general! Doesn't have to be $\mathbb{R}^n$)

A **VECTOR SPACE** is a collection of objects V so that: ANY linear combination of objects in V also belongs to V. That is, if u and v are in V then so is

$$\alpha u + \beta v \quad \text{where } \alpha, \beta \text{ are scalars}$$
(Real #s)

Def - A SUBSPACE V' of V is a subcollection of objects from V which itself forms a vector space (i.e. lin. combs of V' objects are in V')

In order for this to make sense, we actually have to DEFINE "ADDITION" and "SCALING"

- For now, though, let's stick to $\mathbb{R}^n$.
  - $\mathbb{R}^n$ is a vector space! We can certainly take linear combinations of vectors and the result is always in $\mathbb{R}^n$.
  - Today's question: WHAT ARE ALL THE SUBSPACES of $\mathbb{R}^n$?

BASIC PROPERTIES:

- Every vector space at least contains "zero".
- If a vector space contains two points, it also contains the entire line between those two points.

→ SUBSPACES OF $\mathbb{R}^2$:

- Just $\begin{pmatrix}0\\0\end{pmatrix}$!
- Any line through the origin, i.e.

$$ax + by = 0.$$

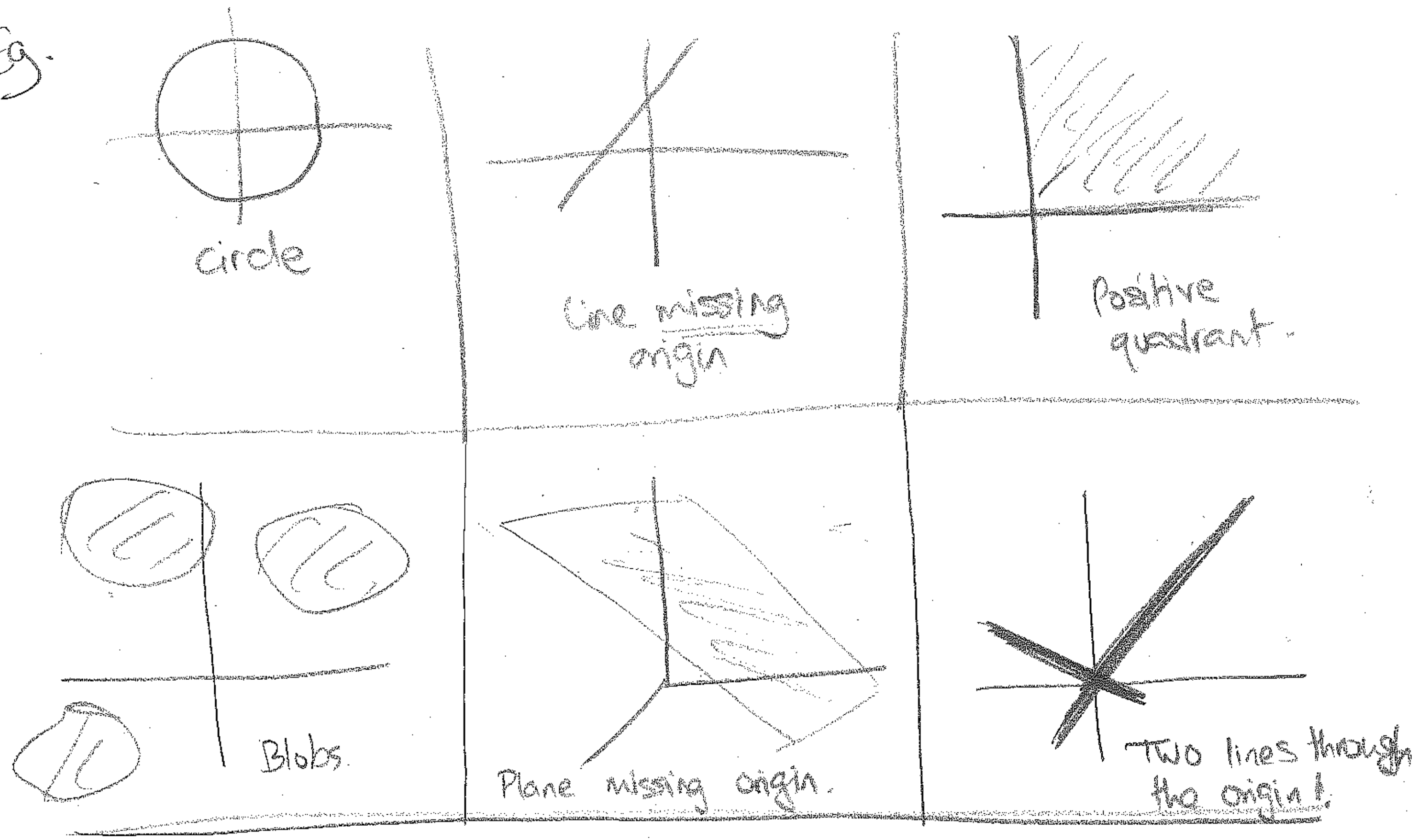
- All of $\mathbb{R}^2$.

→ SUBSPACES OF $\mathbb{R}^3$:

- Just $\begin{pmatrix}0\\0\\0\end{pmatrix}$
- Any line through the origin
- Any plane through the origin
- All of $\mathbb{R}^3$.

What are <u>NOT</u> subspaces?

Eg.



<u>Typical exam problem</u>:

Let $V = \{\text{collection of things}\}$

a) Is V a vector space?

b) Consider $V' = \{\text{subcollection of } V\}$. Is V' a subspace of V?

<u>Eg</u> Let $V = \left\{ \text{matrices } \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \right\}$

a) Is V a vector space?

b) $V' = \left\{ \text{matrices } \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix} \right\}$ Is V' subspace?

c) $V'' = \left\{ \text{matrices } \begin{bmatrix} a & a^3 \\ 0 & -a \end{bmatrix} \right\}$. Is V'' subspace of V'? Of V?

NULL AND COLUMN SPACES : $\boxed{\text{TWO BASIC SUBSPACES associated to a matrix}}$

- Remember, a matrix with m rows and n columns is a <u>FUNCTION</u> from $\mathbb{R}^m$ to $\mathbb{R}^n$.

<u>Eg:</u>

$$\begin{array}{c c} & \overbrace{}^{3 \text{ cols}} \\ 2 \text{ rows} & \begin{bmatrix} 1 & -3 & 2 \\ -2 & 6 & -4 \end{bmatrix} \end{array} = A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

Sends $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to $\begin{bmatrix} x - 3y + 2z \\ -2x + 6y - 4z \end{bmatrix}$

(3D input) (2D output)

<u>Def</u> The NULL SPACE (aka KERNEL) of an $m \times n$ matrix is the collection of ALL vectors in $\mathbb{R}^m$ which it maps to the zero-vector in $\mathbb{R}^n$. (This is a SUBSPACE of $\mathbb{R}^m$)

<u>Def</u> The COLUMN SPACE of an $m \times n$ matrix is the collection of ALL vectors in $\mathbb{R}^n$ that might be output by that matrix. (This is a SUBSPACE of $\mathbb{R}^m$)

PICTURE: